$$h^\circ(t) = \left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right)t - \frac{1}{2}, \qquad 0 \leqslant t < \frac{\sqrt{6}}{2}$$

The optimal control is determined from the maximum principle (2.14)

$$u^\circ(t) = -1, \ 0 \leqslant t \leqslant \sqrt{2}$$
$$u^\circ(t) = 1, \ \sqrt{2} < t \leqslant 2\sqrt{2}$$

The optimal trajectory touches the constraints for $t_1 = \sqrt{6}/2$, $t_2 = 2\sqrt{2} - \sqrt{6}/2$.

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ON THE l-EVASION OF CONTACT IN A LINEAR DIFFERENTIAL GAME

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We derive sufficient conditions for the l-evasion of contact in a linear differential game. The paper adjoins the investigations in [1 – 5].

1. We consider the problem of evasion of contact [1, 2] in a linear differential game [3] given by the equation

$$\dot{z} = Cz + f(u, v), \qquad u \in P, \qquad v \in Q \tag{1.1}$$

Here z is a vector in the n-dimensional Euclidean space R^n, C is a constant nth-order square matrix, u is the pursuit parameter, v is the escape parameter, P and Q are given compact subsets from R^n, $f(u, v)$ is a function continuous in all its variables on $P \times Q$. The terminal set M of game (1.1) is assumed to be a linear subspace of space R^n.

We say that an evasion of contact is possible (or an escape is possible) in game (1.1) if for any initial value $z_0 \in R^n \setminus M$ of the vector z and for an arbitrary measurable variation of the control parameter $u = u(t)$ we can select a measurable variation of the control parameter $v = v(t)$ such that the point $z(t)$, being a solution of the vector differential equation

$$\dot{z} = Cz + f(u(t), v(t)), \qquad z_0 = z(0) \tag{1.2}$$

does not hit onto set M for any value whatsoever of time $t \in (0, +\infty)$. Here, to determine the value of parameter $v(t)$ at each instant t we are allowed to use only the values of $u(s)$ and $z(s)$ for $s \leqslant t$ and we are not allowed to use these values for $s > t$ (see [1]).

We say that an l-evasion of contact (or an l-escape) is possible in game (1.1) if there exist a pair of numbers $\theta \geqslant 0$ and $l > 0$, depending only on the game, such that the control $v = v(t)$ can be constructed on the basis of the information indicated so that the following estimate holds for point $z(t)$:

$$\begin{aligned} \xi(t) &= |\pi z(t)| > 0, \qquad 0 < t \leqslant \theta \\ \xi(t) &= |\pi z(t)| \geqslant l, \qquad \theta \leqslant t < +\infty \end{aligned} \tag{1.3}$$

where π is the operator of orthogonal projection from R^n onto a subspace L which is the orthogonal complement of M in R^n.

2. By $\Phi(t)$ we denote the matrix e^{tC} and by S the unit ball in L. We assume that the following condition has been satisfied for game (1.1).

Condition 1. There exists $\delta > 0$ such that for any $r \in (0, 2\delta]$ and for any $u \in P$ the set

$$\pi\Phi(r) f(u, Q) = w(u, r)$$

is convex, while the set

$$w(r) = \bigcap_{u \in P} w(u, r)$$

has a null vector as an interior point, i.e. there exists $\gamma(r) > 0$ such that

$$\gamma(r) S \subset w(r), \qquad 0 < r \leqslant 2\delta \tag{2.1}$$

(See [3, 4] for the definition of operations over convex sets).

In what follows, by $\gamma(r)$ we shall mean the largest of the numbers satisfying (2.1).

Assertion 1. The function $\gamma(r)$ is continuous and bounded on the interval $(0, 2\delta]$.

Proof. The continuity of a convexly multiple function $w(r)$, $r > 0$ was proved in [3]. Therefore, for any $r_0 \in (0, 2\delta]$ and for any $\varepsilon > 0$ there exists $\eta > 0$ such that the inclusions

$$w(r) \subset w(r_0) + \varepsilon S, \quad w(r_0) \subset w(r) + \varepsilon S \tag{2.2}$$

are fulfilled for any $r \in (r_0 - \eta, r_0 + \eta) \cap (0, 2\delta]$. Together with (2.1) the second one of these inclusions yields the inclusion

$$\gamma(r_0) S \subset w(r) + \varepsilon S$$

Hence (see Assertion 2 in [4]) $\gamma(r_0) S \overset{*}{-} \varepsilon S \subset w(r)$ and, consequently, for any $\varepsilon \leqslant \gamma(r_0)$ we have $(\gamma(r_0) - \varepsilon) S \subset w(r)$ which in accordance with the definition of $\gamma(r)$ yields

$$\gamma(r) \geqslant \gamma(r_0) - \varepsilon \tag{2.3}$$

If, however, $\varepsilon > \gamma(r_0)$, then by virtue of Condition 1 inequality (2.3) is obvious.

Together with (2.1) the first one of inclusions (2.2) yields the inclusion

$$\gamma(r)\, S \subset w(r_0) + \varepsilon S \tag{2.4}$$

Further, from (2.3) it follows that if $\varepsilon < {}^1\!/_2\gamma(r_0)$, then $\gamma(r) > {}^1\!/_2\gamma(r_0) > \varepsilon$, so that from (2.4) we obtain $(\gamma(r) - \varepsilon)\, S \subset w(r_0)$ and, consequently, $\gamma(r) \leqslant \gamma(r_0) + \varepsilon$ for any $r \in (r_0 - \eta,\ r_0 + \eta) \cap (0, 2\delta]$. The boundedness of $\gamma(r)$ follows from the boundedness of $w(u, r)$ for any $u \in P$ (the compactness of Q).

3. By K we denote the unit sphere in L (the boundary of ball S). We assume that the following conditions have been satisfied for game (1.1).

Condition 2. For any $\psi \in K$ there exists a vector $v(\psi) \in Q$ such that

$$(\psi \cdot \Phi(r) f(u, v)) \leqslant (\psi \cdot \Phi(r) f(u, v(\psi))) \tag{3.1}$$

for any $r \in [0, 2\delta]$ and for any $u \in P$, $v \in Q$.

Condition 3. For any $z \in R^n$ there exists a linear subspace $L(z)$ of space L such that

$$\pi\Phi(t) z \in L(z), \qquad 0 \leqslant t \leqslant 2\delta$$

We note that since $\gamma(r)\, S \subset w(u, r)$, it follows instantly from Condition 2 that for any $r \in [0, 2\delta]$ and $u \in P$

$$\gamma(r) \leqslant (\psi \cdot \Phi(r) f(u, v(\psi))) \tag{3.2}$$

Let $z \in R^n$ and let $\psi(z) \in K$ be an arbitrary vector orthogonal to $L(z)$ (it exists by virtue of Condition 3). We fix $\psi(z)$ and we set $v(z) = v(\psi(z))$. Substituting in (3.2) the vector $v(z)$ in the place of $v(\psi)$ and an arbitrary control $u(t - r)$ in the place of u, we obtain by integrating with respect to r from zero to t (see Assertion 1)

$$\left(\psi(z) \cdot \int_0^t \pi\Phi(r) f(u(t - r), v(z))\, dr\right) \geqslant \int_0^t \gamma(r)\, dr = \mu(t), \quad t \in [0, 2\delta] \tag{3.3}$$

4. In [1, 2] it was proved that Condition 1 is sufficient (when $f(u, v) = v - u$) for an escape to be possible in game (1.1). The following theorem answers the question on the possibility of an l-escape.

Theorem. Let Conditions 1 – 3 be fulfilled for game (1.1). Then an l-escape is possible in this game, and $\theta \equiv \delta$, while

$$l = \min_{s \in [0, \delta]} \sqrt{\mu^2(s) + \alpha^2(s)} > 0 \tag{4.1}$$

$$\alpha(s) = \max\{0;\ \mu(s + \delta) - \mu(s) - Ns\} \tag{4.2}$$

$$N = \max|\pi\Phi(r) f(u, v)|, \qquad r \in [0, \delta], \quad u \in P, \quad v \in Q$$

Proof. First of all we note that $\mu(s)$ and $\alpha(s)$ are continuous functions of parameter s, so that the inequalities $\mu(s) > 0$ for $s > 0$ and $\alpha(0) = \mu(\delta) > 0$ guarantee that l is positive. Now let $z_0 = z(0)$ be an arbitrary vector of R^n. For an escape starting from point z_0 we propose to construct inductively the control $v = v(t)$ on each of the intervals $[n\delta, (n + 1)\delta)$, $n = 0, 1, \ldots$, by the rule

$$v(s) \equiv v_n \equiv v(z_n), \qquad s \in [n\delta, (n + 1)\delta) \tag{4.3}$$

where $z_n = z(n\delta)$ is the value of vector $z(t)$ at the instant $n\delta$. Then, according to Cauchy formula, for any $t \in [n\delta, (n + 1)\delta]$

$$\pi z(t) = \pi\Phi(t - n\delta) z_n + \int_0^{t-n\delta} \pi\Phi(r) f(u(t-r), v_n)\, dr \tag{4.4}$$

where $u(s)$, $0 \leqslant s \leqslant t$ is the pursuer's control constructed by the instant t .

By Π_n we denote the orthogonal projection operator from R^n onto $L(z_n)$ and by Γ_n the orthogonal projection operator onto $\psi_n = \psi(z_n)$. By noting that $\Gamma_n x \equiv (\psi_n \cdot x)\psi_n$ for any $x \in L$, from (4.4) we obtain, by virtue of (3.3), the inequality

$$|\Gamma_n \pi z(t)| \geqslant \mu(t - n\delta) \tag{4.5}$$

Since for $n \geqslant 1$

$$\Pi_n \pi z(t) = \pi\Phi(t - n\delta)\left\{\Phi(\delta) z_{n-1} + \int_0^{\delta} \Phi(s) f(u(n\delta - s), v_{n-1})\, ds\right\} + \int_0^{t-n\delta} \Pi_n\Phi(r) f(u(t-r), v_n)\, dr$$

in accordance with the definition of $\psi(z)$ we have

$$\Gamma_{n-1}\Pi_n \pi z(t) = a_n + b_n \tag{4.6}$$

where

$$a_n = \left(\psi_{n-1} \cdot \int_0^{\delta} \Phi(t - n\delta + s) f(u(n\delta - s), v_{n-1})\, ds\right)\psi_{n-1}$$

$$b_n = \int_0^{t-n\delta} \Gamma_{n-1}\Pi_n \pi\Phi(r) f(u(t-r), v_n)\, dr$$

For the first term in (4.6), as also in (3.3), we have the bound (see (3.2))

$$|a_n| \geqslant \int_0^{\delta} \gamma(t - n\delta + s)\, ds = \mu(t - n\delta + \delta) - \mu(t - n\delta) \tag{4.7}$$

The estimate

$$|b_n| \leqslant N \cdot (t - n\delta) \tag{4.8}$$

is obvious for the second term in (4.6). From (4.7), (4.8) it follows that (see (4.2))

$$|\Pi_n \pi z(t)| \geqslant |\Gamma_{n-1}\Pi_n \pi z(t)| \geqslant \alpha(t - n\delta)$$

Hence, finally, for any $t \in [n\delta, (n+1)\delta)$, $n \geqslant 1$.

$$\xi^2(t) = |\pi z(t)|^2 = |\Pi_n \pi z(t)|^2 + |\Gamma_n \pi z(t)|^2 \geqslant \mu^2(t - n\delta) + \alpha^2(t - n\delta) \geqslant l^2$$

For $n = 0$, from (4.5) we have

$$\xi(t) \geqslant |\Gamma_0 \pi z(t)| \geqslant \mu(t) > 0, \qquad t \in (0, \delta)$$

Q.E.D.

5. Let us consider the escape game "the boy and the crocodile" [1]

$$z^{\cdot 1} = z^2 + v, \quad z^{\cdot 2} = -u, \quad |u| \leqslant 1, \quad |v| \leqslant 1 \tag{5.1}$$

Here z^1, z^2, u, v are ν-dimensional vectors of the Euclidean space R^ν, u and v are the control parameters. The terminal set M consists of those and only those points

$z = (z^1, z^2)$ for which $z^1 = 0$. In this connection $L = \{z : z^2 = 0\}$ and $\pi z = z^1$ (the second coordinate, equalling zero, is omitted). We assume that $\nu \geqslant 3$.

In this problem we can obtain a bound for the quantity l somewhat better than the bound (4.1) given by the Theorem in Sect. 4. Namely, in spite of the fact that Condition 1 is not satisfied on the interval [0, 2], for an escape starting from the point $z_0 = (z_0^1, z_0^2)$ we propose to construct inductively the control $v = v(t)$ on each of the intervals $I_n = [n, n+1)$, $n = 0, 1, 2, \ldots$, by the rule $v(s) \equiv v_n$, $s \in I_n$, where v_n is the unit vector from R^ν orthogonal to the vectors $z^1(n)$ and $z^2(n)$ (here $z^1(n)$ and $z^2(n)$ are the values of vectors $z^1(t)$ and $z^2(t)$ at the instant n). Then, by virtue of the Cauchy's formula

$$\pi z(t) \equiv z^1(t) = z^1(n) + (t-n) z^2(n) + (t-n) v_n - \int_0^{t-n} r u(t-r)\, dr, \quad t \in I_n \tag{5.2}$$

where $u(s)$, $0 \leqslant s \leqslant t$ is the pursuer's control constructed by the instant t.

Denoting as before by Π_n the orthogonal projection operator from L onto the subspace spanned by the vectors $z^1(n)$ and $z^2(n)$, and by Γ_n the orthogonal projection operator onto v_n, we have

$$|\Gamma_n z^1(t)| \geqslant (t-n) - \int_0^{t-n} r\, dr = \mu(t-n), \quad t \in I_n \tag{5.3}$$

$$\mu(s) = s - {}^1/_2 s^2, \qquad 0 \leqslant s \leqslant 2$$

Since for $n \geqslant 1$

$$\Pi_n z^1(t) = z^1(n-1) + (1+t-n) z^2(n-1) + v_{n-1} - \int_0^{t-n} r \Pi_n u(t-r)\, dr - \int_0^1 r u(n-r)\, dr - \int_0^1 (t-n) u(n-r)\, dr$$

we have

$$|\Pi_n z^1(t)| \geqslant |\Gamma_{n-1} \Pi_n z^1(t)| \geqslant 1 - {}^1/_2 (t-n)^2 - {}^1/_2 - (t-n) \tag{5.4}$$

From inequalities (5.3) and (5.4) we have

$$|z^1(t)|^2 \geqslant \mu^2(t-n) + \alpha^2(t-n), \quad t \in I_n, \quad n \geqslant 1$$

$$\alpha(s) = \max\{0;\ {}^1/_2 - s - {}^1/_2 s^2\}, \ 0 \leqslant s \leqslant 1$$

Computing the minimum of the function $\mu^2(s) + \alpha^2(s)$ on the interval [0, 1], for the evasion magnitude we obtain the value $l_0 = 0.2978$.

Now, let the initial point $z_0 = (z_0^1, z_0^2)$ be such that

$$|z_0^1 + t z_0^2|^2 \geqslant (l_0 + {}^1/_2 t^2)^2 - t^2, \quad 0 \leqslant t \leqslant t_0, \quad t_0 = 1 - \sqrt{1 - 2l_0}$$

where t_0 is the smallest positive root of the equation $l_0 = t - {}^1/_2 t^2$. Then the indicated behavior of the escaper guarantees l_0-evasion of contact, starting from the instant $t = 0$. In fact, by virtue of (5.2)

$$|\pi z(t)| \geqslant (|z_0^1 + t z_0^2|^2 + t^2)^{1/2} - {}^1/_2 t^2 \geqslant l_0, \quad 0 \leqslant t \leqslant t_0$$

For $t \in [t_0, 1]$ the required estimate follows from (5.3).

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INVESTIGATION OF CERTAIN OPTIMAL SYSTEMS BY THE AVERAGING METHOD

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We construct the canonic averaging scheme for solving certain optimal control problems on the basis of Pontriagin maximum principle. We assume that the plant is described by a system with rotating phase [1], while the control enters only into the perturbing terms [2]. The analysis is carried out on a large time interval so that the controlled quantities vary significantly. The procedure developed is illustrated by concrete examples of quasi-linear oscillatory systems. The small parameter method for the approximate solution of optimal control problems was employed in [2 – 5].

1. Statement of the problem. We formulate the problem of controlling a certain mechanical plant by small control actions. Let the corresponding system of equations have the form

$$x^{\cdot} = \varepsilon X(\tau, x, y, u, \varepsilon), \quad \tau = \varepsilon(t - t_0) + \tau_0, \quad x(t_0) = x_0 \qquad (1.1)$$
$$y^{\cdot} = Y_0(\tau, x, y) + \varepsilon Y(\tau, x, y, u, \varepsilon), \qquad y(t_0) = y_0$$

Here x, X are n-dimensional vectors; y, Y_0, Y are m-dimensional vectors; u is the l-dimensional control, τ is "slow time", ε is a small scalar parameter, $\varepsilon \in [0, \varepsilon_0]$. We assume that the right-hand sides of system (1.1) have been defined in some, possibly unbounded, region of variation of their arguments and in it satisfy all the necessary smoothness and periodicity conditions which follow from the subsequent constructions. The control's performance criterion will be introduced somewhat later, after the derivation of a standard system with rotating phase.

From (1.1) it follows that when $\varepsilon = 0$ the system becomes uncontrollable